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Constructing Green's function for the time-dependent Maxwell system in anisotropic dielectrics

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Abstract

A method of constructing explicit formulae for Green's matrix function of the time-dependent Maxwell system for a homogeneous non-dispersive dielectric with a general form of the anisotropy is described. This method consists of two parts. In the first part a matrix function depending on the time variable and three Fourier parameters is determined. This matrix function, being the Fourier image of Green's matrix function with respect to three space variables, is constructed in an explicit form. The second part of the method is the inverse Fourier transform of the determined matrix function. The robustness and advantages of this method are confirmed by numerical experiments. Examples of simulations of electromagnetic fields in anisotropic crystals are presented.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Finding Green's functions of partial differential equations and systems describing different physical processes and phenomena is one of the fundamental topics in mathematical physics. The essential ingredient in the boundary element method for modelling mechanical and electromagnetic behaviour of advanced materials is Green's function as well [1–5]. Green's functions for many scalar equations of mathematical physics with constant coefficients have been found in an explicit form (see, for example [6–8]). Explicit formulae for Green's functions for isotropic linear elasticity and electrodynamics in infinite homogeneous media are well known (see, for example [3, 9]). Green's function study for anisotropic elastic materials is contained in [10–12]. Different approaches to constructing Green's function for the Maxwell system in isotropic and particular cases of anisotropic media were studied in [13–20]. But besides that it would be very useful to know Green's function for anisotropic electromagnetic media with a general form of the anisotropy for the study of such advanced materials as anisotropic dielectrics, ferrites and plasmas.

This paper is devoted to constructing Green's function for time-dependent electromagnetic fields in homogeneous non-dispersive anisotropic dielectrics. Maxwell's system is considered in infinite space with zero initial data and directional pulse point density of an electric current with three different basis directions. The main result of this paper is a new method of constructing explicit formulae for Green's function of Maxwell's system for crystals with a general form of the anisotropy. This method consists of two parts. In the first part a matrix function depending on a time variable and three parameters is constructed. This matrix, being the Fourier image of Green's function with respect to three space variables, is constructed in an explicit form. In the second part Green's function is found by the inverse Fourier transform of this matrix function. The robustness and advantages of this method are confirmed by computational experiments.

The paper is organized as follows. The equations of electrodynamics and the notion of Green's function as a matrix are described in section 2. Finding Green's function of the electric field in dielectrics is addressed in section 3. Analysis of computational steps and examples of simulations electromagnetic fields in anisotropic crystals are described in section 4. Section 5 contains a conclusion and remarks.

2. Green's function for Maxwell's system

Let us consider an electromagnetic medium which is ideal, homogeneous, non-dispersive, and linear, but anisotropic [21]. Anisotropic dielectrics can be considered as an example of such anisotropic media. The dynamic wave propagation of electromagnetic waves in this medium is described by the time-dependent system in which the dielectric permittivity is given by a 3×3 matrix [21].

Let $x = (x_1, x_2, x_3)$ be a space variable from R^3, t be a time variable from R, then Maxwell's equations are

$$\mathrm{curl}_x \mathbf{H} = \mathcal{E}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}, \qquad \mathrm{curl}_x \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \tag{1}$$

$$\mathrm{div}_x(\mathcal{E}\mathbf{E}) = \rho, \qquad \mathrm{div}_x(\mu\mathbf{H}) = 0, \tag{2}$$

where $\mathbf{E} = (E_1, E_2, E_3), \mathbf{H} = (H_1, H_2, H_3)$ are electric and magnetic intensity vectors, $E_k = E_k(x, t), H_k = H_k(x, t), k = 1, 2, 3; \mathbf{j} = (j_1, j_2, j_3)$ is the density of the electric current, $j_k = j_k(x, t), k = 1, 2, 3; \mu$ is the magnetic permeability, $\mathcal{E}$ is the dielectric permittivity, ρ is the density of electric charges. We assume the conservation law of charges

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_x \mathbf{j} = 0 \tag{3}$$

is given and $\mu = 1$, $\mathcal{E} = (\epsilon_{ij})_{3\times 3}$ is a symmetric positive definite matrix with constant elements.

We also suppose that

$$\mathbf{E} = 0, \qquad \mathbf{H} = 0 \qquad \text{for} \quad t \leqslant 0, \tag{4}$$

and

$$\rho = 0, \qquad \mathbf{j} = 0 \qquad \text{for} \quad t \leqslant 0.$$

This means there is no electromagnetic field, currents, or electric charges at the time $t < 0$.

Remark 1. *We note that the second equation of (2) follows from the second equation of (1) and (4). The first equation of (2) can be obtained from the first equation of (1), equation (3) and conditions (4). It means that equations (1) under conditions (4) are principal but conditions (2) are consequences of (1), (3), (4).*

Let $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{H} = (H_1, H_2, H_3)$ be vector functions satisfying

$$\mathrm{curl}_x \mathbf{H} = \mathcal{E}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{e}\delta(x)\delta(t), \tag{5}$$

$$\mathrm{curl}_x \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \tag{6}$$

$$\mathbf{E}|_{t\leqslant 0} = 0, \qquad \mathbf{H}|_{t\leqslant 0} = 0, \tag{7}$$

where $\mathbf{e}$ is an arbitrary unit vector from R^3; $\delta(t)$ is the Dirac delta function with respect to t; $\delta(x) = \delta(x_1)\delta(x_2)\delta(x_3)$ is the Dirac delta function with respect to space variables. Let further $\mathbf{E}^n = \left(E_1^n, E_2^n, E_3^n\right)$, $\mathbf{H}^n = \left(H_1^n, H_2^n, H_3^n\right)$ be vector functions satisfying (5)–(7) for $\mathbf{e} = \mathbf{e}^n$, where $n = 1, 2, 3$; $\mathbf{e}^1 = (1, 0, 0)$, $\mathbf{e}^2 = (0, 1, 0)$, $\mathbf{e}^3 = (0, 0, 1)$.

A matrix $\mathcal{G}(x, t) = (G_{mn}(x, t))_{6\times 3}$ is a Green's function for Maxwell's system if elements of this matrix are defined as follows:

$$\begin{aligned} G_{mn} &= E_m^n, & m &= 1, 2, 3; & n &= 1, 2, 3; \\ G_{mn} &= H_{m-3}^n, & m &= 4, 5, 6; & n &= 1, 2, 3, \end{aligned}$$

where E_m^n, $m = 1, 2, 3$ are components of $\mathbf{E}^n$, and H_{m-3}^n, $m = 4, 5, 6$ are components of $\mathbf{H}^n$.

To construct Green's function means to solve the problem (5)–(7) for $\mathbf{e} = \mathbf{e}^n$, $n = 1, 2, 3$. We also note that differentiating (5) with respect to t and using (6) we find that the electric field $\mathbf{E}$ has to satisfy the vector equation

$$-\mathrm{curl}_x \mathrm{curl}_x \mathbf{E} = \mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mathbf{e}\delta(x)\delta'(t), \qquad x \in R^3, \quad t \in R, \tag{8}$$

$$\mathbf{E}|_{t\leqslant 0} = 0, \tag{9}$$

where $\mathbf{e}$ is an arbitrary unit vector from R^3; $\delta'(t)$ is the derivative of the Dirac delta function $\delta(t)$.

In this paper we use equation (8) and condition (9) to construct $\mathbf{E}$, and consequently $\mathbf{E}^n$ if $\mathbf{e} = \mathbf{e}^n$, $n = 1, 2, 3$. After finding $\mathbf{E}$ and $\mathrm{curl}_x\mathbf{E}$ (or $\mathbf{E}^n$ and $\mathrm{curl}_x\mathbf{E}^n$, $n = 1, 2, 3$) we use equation (6) and condition (7) to find $\mathbf{H}$ (or $\mathbf{H}^n$, $n = 1, 2, 3$) by solving the ordinary differential equation (6) with zero initial data.

3. Constructing Green's function of the electric field

The main object of this section are problems (8), (9). The main outcome here is the method of this problem solving for any unit vector $\mathbf{e}$. As a result Green's function of the electric field is constructed.

Green's function of the electric field here is a matrix $\mathcal{G}_E = \left(E_m^n\right)_{3\times 3}$ whose nth column $\mathbf{E}^n = \left(E_1^n, E_2^n, E_3^n\right)$ satisfies (8), (9) for $\mathbf{e} = \mathbf{e}^n$, $n = 1, 2, 3$.

Let $\tilde{\mathbf{E}}(\nu, t)$ be the Fourier transform image of the electric field $\mathbf{E}(x, t)$ with respect to the space variable $x = (x_1, x_2, x_3) \in R^3$, i.e.

$$\tilde{\mathbf{E}}(\nu, t) = \mathcal{F}_x[\mathbf{E}](\nu, t),$$

where

$$\mathcal{F}_x[\mathbf{E}](\nu, t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{E}(x, t)\,\mathrm{e}^{\mathrm{i}\nu x}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3,$$

$$\nu = (\nu_1, \nu_2, \nu_3), \qquad x\nu = x_1\nu_1 + x_2\nu_2 + x_3\nu_3, \qquad \mathrm{i}^2 = -1.$$

Problems (8), (9) can be written in terms of the Fourier image $\tilde{\mathbf{E}}(\nu, t)$ as follows:

$$\mathcal{E}\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2} + \mathcal{S}(\nu)\tilde{\mathbf{E}} = -\mathbf{e}\delta'(t), \qquad t \in R, \quad \nu \in R^3. \tag{10}$$

$$\tilde{\mathbf{E}}|_{t\leqslant 0} = 0, \qquad \nu \in R^3, \tag{11}$$

where

$$\mathcal{S}(\nu) = \begin{pmatrix} \nu_2^2 + \nu_3^2 & -\nu_1\nu_2 & -\nu_1\nu_3 \\ -\nu_1\nu_2 & \nu_1^2 + \nu_3^2 & -\nu_2\nu_3 \\ -\nu_1\nu_3 & -\nu_2\nu_3 & \nu_1^2 + \nu_2^2 \end{pmatrix}. \tag{12}$$

A procedure of solving (10), (11) is described below. The starting point for solving problem (10), (11) is the construction of a non-singular matrix $\mathcal{T}$ such that

$$\mathcal{T}^T(\nu)\mathcal{E}\mathcal{T}(\nu) = \mathcal{I}, \tag{13}$$

$$\mathcal{T}^T(\nu)\mathcal{S}(\nu)\mathcal{T}(\nu) = \mathcal{D}(\nu), \tag{14}$$

where $\mathcal{I}$ is the identity matrix, $\mathcal{D}(\nu) = \mathrm{diag}(d_1(\nu), d_2(\nu), d_3(\nu))$, $d_k(\nu) \geqslant 0, k = 1, 2, 3$; $\mathcal{T}^T(\nu)$ is the transposed matrix to $\mathcal{T}(\nu)$. The matrix $\mathcal{T}(\nu)$ with properties (12), (13) exists according to the matrix theory, see, for example, [22]. The scheme of this matrix construction is the following. Using the diagonalization matrix procedure for the matrix $\mathcal{E}$ we find an orthogonal matrix $\mathcal{P}$ and a diagonal matrix $\mathcal{M}$ such that

$$\mathcal{P}^T\mathcal{E}\mathcal{P} = \mathcal{M},$$

where $\mathcal{M} = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$, $\mu_k > 0, k = 1, 2, 3$.

Remark 2. *We note here that $\mu_k > 0, k = 1, 2, 3$ are eigenvalues of $\mathcal{E}$. Let $\mathcal{M}^{\frac{1}{2}}$ be defined by*

$$\mathcal{M}^{\frac{1}{2}} = \mathrm{diag}(\sqrt{\mu_1}, \sqrt{\mu_2}, \sqrt{\mu_3}),$$

then $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}$ are defined by

$$\mathcal{E}^{\frac{1}{2}} = \mathcal{P}\mathcal{M}^{\frac{1}{2}}\mathcal{P}^T, \qquad \mathcal{E}^{-\frac{1}{2}} = \left(\mathcal{E}^{\frac{1}{2}}\right)^{-1}.$$

Remark 3. *The matrices $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}$ satisfy the following properties: $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}$ have constant elements; $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}$ are positive symmetric matrices;*

$$\left(\mathcal{E}^{-\frac{1}{2}}\right)^{-1} = \mathcal{E}^{\frac{1}{2}}; \qquad \mathcal{E}^{-\frac{1}{2}}\mathcal{E} = \mathcal{E}^{\frac{1}{2}}; \qquad \left(\mathcal{E}^{-\frac{1}{2}}\right)^T = \mathcal{E}^{-\frac{1}{2}}.$$

Consider now the matrix $\mathcal{E}^{-\frac{1}{2}}\mathcal{S}(\nu)\mathcal{E}^{-\frac{1}{2}}$, where $\mathcal{S}(\nu)$ is defined by (12), and $\mathcal{E}^{-\frac{1}{2}}$ is found by the above-mentioned procedure. Using the diagonalization matrix procedure for the matrix $\mathcal{E}^{-\frac{1}{2}}\mathcal{S}(\nu)\mathcal{E}^{-\frac{1}{2}}$ we find an orthogonal matrix $\mathcal{Q}(\nu)$ and a diagonal matrix $\mathcal{D}(\nu)$ such that

$$\mathcal{D}(\nu) = \mathcal{Q}^T(\nu)\left[\mathcal{E}^{-\frac{1}{2}}\mathcal{S}(\nu)\mathcal{E}^{-\frac{1}{2}}\right]\mathcal{Q}(\nu), \tag{15}$$

where $\mathcal{D}(\nu) = \mathrm{diag}(d_1(\nu), d_2(\nu), d_3(\nu))$, $d_k(\nu) \geqslant 0, k = 1, 2, 3$; $\mathcal{Q}^T(\nu) = \mathcal{Q}^{-1}(\nu)$.

Letting now $\mathcal{T}(\nu) = \mathcal{E}^{-\frac{1}{2}}\mathcal{Q}(\nu)$ we have the matrix $\mathcal{T}(\nu)$ which satisfies (13), (14), where the matrix $\mathcal{D}(\nu)$ is defined by (15).

We are looking for the solution of problem (10), (11) in the form

$$\tilde{\mathbf{E}}(\nu, t) = \mathcal{T}(\nu)\mathbf{Y}(\nu, t), \tag{16}$$

where the matrix $\mathcal{T}(\nu)$ is just constructed and a vector function $\mathbf{Y}(\nu, t)$ is unknown. Substituting (16) into (10), (11) we find

$$\mathcal{E}\mathcal{T}\frac{\mathrm{d}^2\mathbf{Y}}{\mathrm{d}t^2} + \mathcal{S}(\nu)\mathcal{T}\mathbf{Y} = -\mathbf{e}\delta'(t), \qquad t \in R, \quad \nu \in R^3, \tag{17}$$

$$\mathbf{Y}(\nu, t)|_{t\leqslant 0} = 0, \qquad \nu \in R^3. \tag{18}$$

Multiplying (17) by $\mathcal{T}^T(\nu)$ and using (13), (14) we have

$$\frac{\mathrm{d}^2\mathbf{Y}}{\mathrm{d}t^2} + \mathcal{D}(\nu)\mathbf{Y} = -\mathcal{T}^T(\nu)\mathbf{e}\delta'(t), \qquad t \in R, \quad \nu \in R^3. \tag{19}$$

The solution of the Cauchy problem (19), (18) is given by

$$\mathbf{Y}(\nu, t) = \mathrm{column}(Y_1(\nu, t), Y_2(\nu, t), Y_3(\nu, t)), \tag{20}$$

where for $t < 0$ the function $Y_n(\nu, t)$ vanishes and for $t \geqslant 0$ it is defined by

$$Y_n(\nu, t) = -\begin{cases} [\mathcal{T}^T(\nu)\mathbf{e}]_n \cos(\sqrt{d_n(\nu)}t), & d_n(\nu) > 0 \\ [\mathcal{T}^T(\nu)\mathbf{e}]_n, & d_n(\nu) = 0 \end{cases} \qquad (n = 1, 2, 3).$$

The solution of (10), (11) is given now by formula (16), where the vector function $\mathbf{Y}(\nu, t)$ is defined by (20). The pre-image $\mathbf{E}(x, t)$ is determined by the inverse Fourier transform of $\tilde{\mathbf{E}}(\nu, t)$.

4. Computational steps and examples of simulations

4.1. Computational steps

The computation of explicit presentations for the matrices $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}, \mathcal{Q}(\nu), \mathcal{Q}^T(\nu)\mathcal{T}(\nu)$, $\mathcal{T}^T(\nu), \mathcal{D}(\nu)$ was realized by the symbolic transformation in MATLAB [23]. Formulae obtained for these matrices are cumbersome in the case of an arbitrary positive definite matrix $\mathcal{E}$. For the case when $\mathcal{E} = \varepsilon\mathbf{I}$, the matrices $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}, \mathcal{Q}(\nu), \mathcal{T}(\nu), \mathcal{D}(\nu)$ have the following forms:

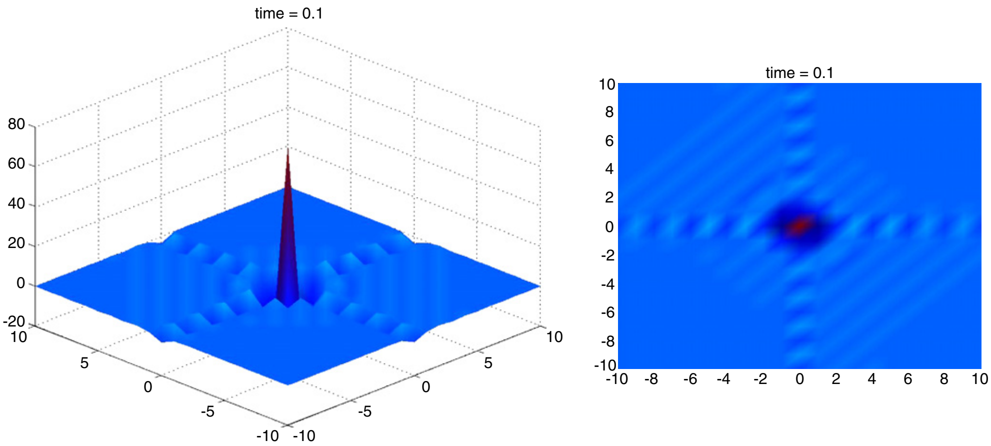
$$\mathcal{E}^{\frac{1}{2}} = \mathrm{diag}(\sqrt{\varepsilon}, \sqrt{\varepsilon}, \sqrt{\varepsilon}), \qquad \mathcal{E}^{-\frac{1}{2}} = \mathrm{diag}\left(\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}\right),$$

$$\mathcal{Q}(\nu) = \begin{pmatrix} \nu_1|\nu|^{-1} & -\nu_3 a^{-1} & -\nu_1\nu_2(a|\nu|)^{-1} \\ \nu_2|\nu|^{-1} & 0 & a|\nu|^{-1} \\ \nu_3|\nu|^{-1} & \nu_1 a^{-1} & -\nu_2\nu_3(a|\nu|)^{-1} \end{pmatrix},$$

$$\mathcal{D}(\nu) = \mathrm{diag}\left(0, \frac{|\nu|^2}{\varepsilon}, \frac{|\nu|^2}{\varepsilon}\right), \qquad \mathcal{T}(\nu) = \frac{1}{\sqrt{\varepsilon}}\mathcal{Q}(\nu),$$

here $|\nu|^2 = \nu_1^2 + \nu_2^2 + \nu_3^2$, $a = \sqrt{\nu_1^2 + \nu_3^2}$.

We note that explicit formulae for matrices $\mathcal{Q}(\nu), \mathcal{D}(\nu)$ and $\mathcal{T}(\nu)$ take several pages even if $\mathcal{E} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_k > 0$, $k = 1, 2, 3$. The sequence of MATLAB commands for finding these matrices is listed in the appendix of this paper.



(*a*) (*b*)

Figure 1. The initial state: (*a*) 3D level plot of $E_1(x_1, x_2, 1, 0)$ and (*b*) 2D level plot of $E_1(x_1, x_2, 1, 0)$.

As a result from explicit formulae for $\mathcal{E}^{\frac{1}{2}}, \mathcal{E}^{-\frac{1}{2}}$, $\mathcal{Q}(\nu)$, $\mathcal{Q}^T(\nu)$, $\mathcal{T}(\nu)$, $\mathcal{T}^T(\nu)$, $\mathcal{D}(\nu)$ and (16), (20) we obtained the explicit form for the image $\tilde{\mathbf{E}}(\nu, t)$ of the electric field.

In the next step we have to calculate 3D inverse Fourier transform of $\tilde{\mathbf{E}}(\nu, t)$ with respect to $\nu = (\nu_1, \nu_2, \nu_3)$. Because of the complexity of the explicit form for $\tilde{\mathbf{E}}(\nu, t)$ the symbolic calculation of the inverse Fourier transform was unsuccessful. That was the reason why the numerical calculation of the inverse Fourier transform was realized on this step.

4.2. Examples of simulations

In this subsection we present the images of the wave propagations in anisotropic dielectrics belonging to different crystal systems [21]. These pictures are obtained by fixing one of the space variables in the component $E_1(x, t)$ of the solution $\mathbf{E}(x, t)$ of (3), (4). For experiments we took the current density in the form $\mathbf{j} = \mathbf{e}\delta(x)\delta(t)$, where $\mathbf{e} = \mathbf{e}^1$. This pulse electric source is concentrated at the point (0, 0, 0).

The electric permittivity $\mathcal{E}$ for different dielectrics is given by the following matrices:

$$\mathcal{E} = \mathrm{diag}(1.07, 1.07, 1.07) \quad \text{(barium peroxide, BaO}_2\text{, [24])}$$

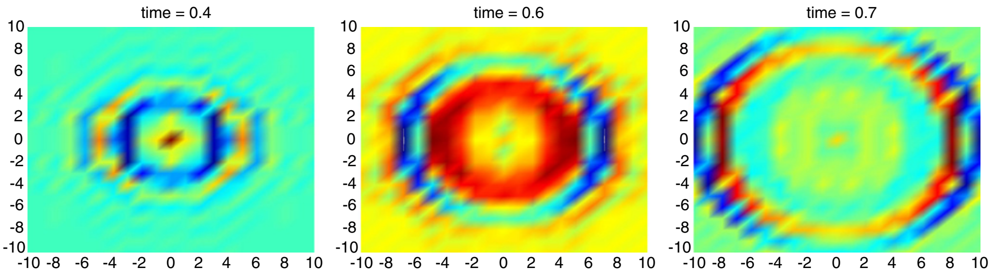
$$\mathcal{E} = \mathrm{diag}(1.8, 1.8, 3.25) \quad \text{(mercurous sulfide, HgS, [24])}$$

$$\mathcal{E} = \mathrm{diag}(1.64, 2.09, 3.24) \quad \text{(magnesium niobate, MgNb}_2\text{O}_6\text{, [24])}$$

$$\mathcal{E} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \quad \text{(a dielectric with the monoclinic anisotropy)}$$

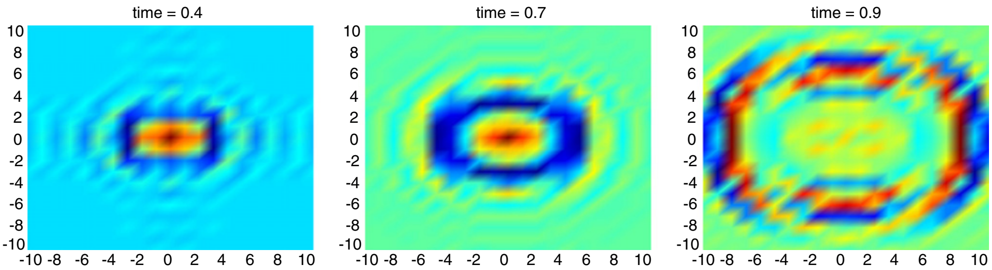
$$\mathcal{E} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 4 \\ 2 & 4 & 9 \end{pmatrix}, \quad \text{(a dielectric with the triclinic anisotropy).}$$

Figure 1 contains the visualization of the component $E_1(x_1, x_2, 1, t)$ of the solution $E(x, t)$ for $t \cong 0$. Figure 1(*a*) is a 3D graph of $E_1(x_1, x_2, 1, 0)$. On the vertical axis are plotted values



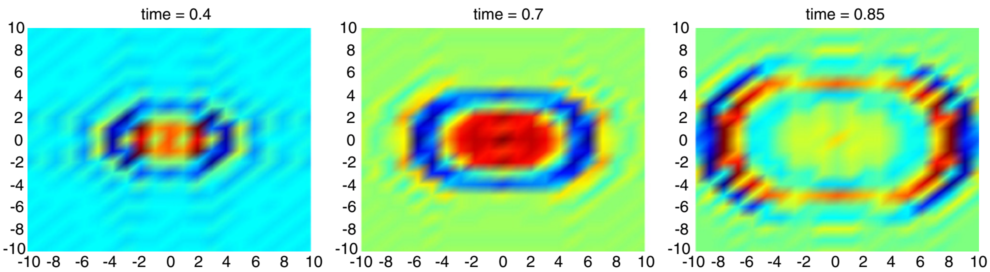
(a) $t = 0.4$ (b) $t = 0.6$ (c) $t = 0.7$

Figure 2. Wave propagation in barium peroxide: $E_1(x_1, x_2, 1, t)$.



(a) $t = 0.4$ (b) $t = 0.7$ (c) $t = 0.9$

Figure 3. Wave propagation in mercurous sulfide: $E_1(x_1, x_2, 1, t)$.

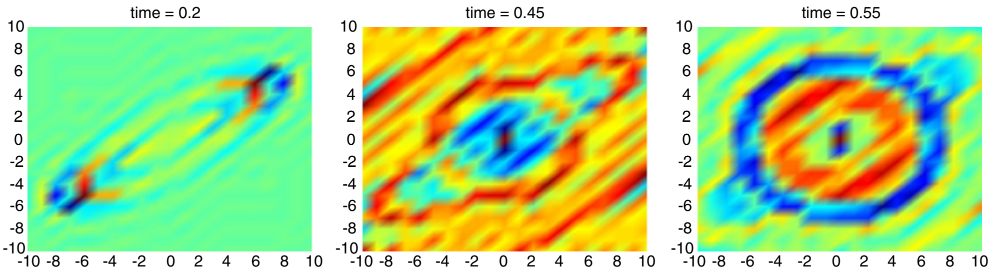


(a) $t = 0.4$ (b) $t = 0.7$ (c) $t = 0.85$

Figure 4. Wave propagation in magnesium niobate: $E_1(x_1, x_2, 1, t)$.

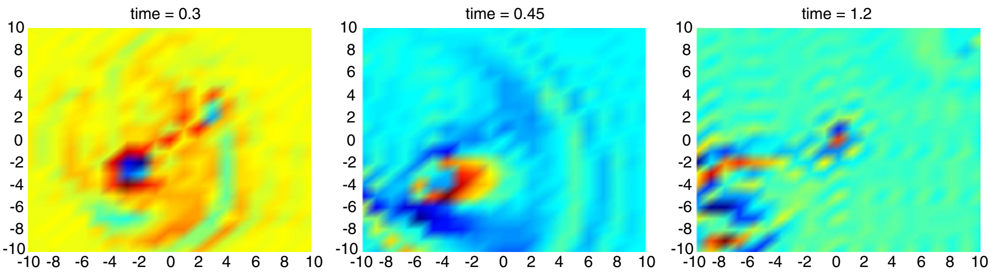
of E_1, the horizontal axes are x_1, x_2. Figure 1 (b) is two-level plots of the same surface of $E_1(x_1, x_2, 1, 0)$. We note that graphs of $E_1(x_1, x_2, 1, 0)$ for different dielectrics are similar for $t = 0$.

Figures 2–4 contain three screen shots of the wave propagations in dielectrics barium peroxid (BaO_2), mercurous sulfide (HgS), magnesium niobate ($MgNb_2O_6$), respectively. The matrices of electric permittivities $\mathcal{E}$ have the diagonal form for these crystals. Figures 5 and 6 contain screen shots of waves in dielectrics with monoclinic and triclinic structures of the anisotropy. The permittivity matrices $\mathcal{E}$ are not diagonal in these cases. These figures are 2D level plots of $E_1(x_1, x_2, 1, t)$ for the different time.



$(a)\ t = 0.2$ $(b)\ t = 0.45$ $(c)\ t = 0.55$

Figure 5. Wave propagation in a dielectric with the monoclinic anisotropy: $E_1(x_1, x_2, 1, t)$.



$(a)\ t = 0.2$ $(b)\ t = 0.45$ $(c)\ t = 0.55$

Figure 6. Wave propagation in a dielectric with the triclinic anisotropy: $E_1(x_1, x_2, 1, t)$.

5. Conclusion and remarks

An efficient method for constructing Green's function of the time-dependent Maxwell system in dielectrics with a general form of anisotropy was described in this paper. The very important step in this method was finding the explicit form of the Fourier image of Green's function with respect to the space variables. After that Green's function was found by the inverse Fourier transform of this explicit image. The second step was obtained by the numerical calculation. The robustness of this method was confirmed by several computational experiments.

We note that the explicit formulae for the Fourier image of Green's function for the electric field $\tilde{\mathcal{G}}_E(\nu, t)$ can be efficiently used for finding electric field arising from an arbitrary electric current density. To show this fact let us suppose now that $\mathbf{j}(x, t)$ be an arbitrary vector function with the explicit form for the Fourier image $\tilde{\mathbf{j}}(\nu, t) = \mathcal{F}_x[\mathbf{j}](\nu, t)$; $\mathbf{E}(x, t)$ be the electric field satisfying the following vector equation:

$$-\mathrm{curl}_x \mathrm{curl}_x \mathbf{E} = \mathcal{E} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\partial \mathbf{j}}{\partial t}, \qquad x \in R^3, \quad t \in R,$$

$$\mathbf{E}|_{t \leqslant 0} = 0.$$

Let further

$$\tilde{\mathbf{E}}(\nu, t) = \mathcal{F}_x[\mathbf{E}](\nu, t), \qquad \tilde{\mathcal{G}}_E(\nu, t) = \mathcal{F}_x[\mathcal{G}_E](\nu, t), \qquad \nu = (\nu_1, \nu_2, \nu_3).$$

Using the generalized function technique [7] we can show that the expression for $\tilde{\mathbf{E}}(\nu, t)$ is calculated by the following formula:

$$\tilde{\mathbf{E}}(\nu, t) = \int_0^\infty \tilde{\mathcal{G}}_E(\nu, t - \tau) \tilde{\mathbf{j}}(\nu, \tau)\, \mathrm{d}\tau.$$

Using this formula and the procedure of section 3 the expressions for $\tilde{\mathcal{G}}_E(\nu, t-\tau)$, $\tilde{\mathbf{E}}(\nu, t)$ can be found in explicit forms by symbolic transformations. To find $\mathbf{E}(x, t)$ as the next step we need to apply the inverse Fourier transform numerically to $\tilde{\mathbf{E}}(\nu, t)$ with respect to three parameters ν_1, ν_2, ν_3.

We note that the simulation of electromagnetic fields based on explicit formulae is the best one. But unfortunately it is impossible to find explicit formulae for general inhomogeneous anisotropic media. For this we need to construct approximate solutions by numerical procedures and methods and then simulate electromagnetic fields. The method of this paper for constructing Green's function in anisotropic homogeneous media can be used as a staring point for finding Green's functions in anisotropic inhomogeneous media for small values of t at a small neighbourhood of the point $x = 0$, where the pulse point source is concentrated. In the case of a small neighbourhood of the point $x = 0$ we can suppose that the inhomogeneous medium is approximated by the homogeneous medium.

Acknowledgments

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Appendix

This section deals with a description of a procedure for finding matrices $\mathcal{E}^{\frac{1}{2}}$, $\mathcal{E}^{-\frac{1}{2}}$, $\mathcal{Q}(\nu)$, $\mathcal{Q}^T(\nu)$, $\mathcal{D}(\nu)$, $\mathcal{T}(\nu)$, $\mathcal{T}^T(\nu)$ by MATLAB. The list of commands of matrix operations is the following:

$$
\begin{aligned}
&\mathit{INPUT} : \mathit{Eps}, S\\
&[\mathit{EigVecEps},\ \mathit{EigValEps}] = \mathit{eig}(\mathit{Eps})\\
&P = \mathit{EigVecEps}\\
&PT = \mathit{Inv}(\mathit{EigVecEps})\\
&M = \mathit{EigValEps}\\
&Mh = \mathit{sqrt}(M)\\
&\mathit{SqrEps} = P * Mh * PT\\
&\mathit{InvSrtEps} = \mathit{inv}(\mathit{SqrEps})\\
&A = \mathit{simplify}(\mathit{InvSrtEps} * S * \mathit{InvSrtEps})\\
&[\mathit{EigVecA},\ \mathit{EigValA}] = \mathit{eig}(A)\\
&D = \mathit{simplify}(\mathit{EigValA})\\
&Q = \mathit{simplify}(\mathit{EigVecA})\\
&Q(:,1) = Q(:,1)./\mathit{sqrt}(\mathit{sum}(Q(:,1).\hat{}2))\\
&Q(:,2) = Q(:,2)./\mathit{sqrt}(\mathit{sum}(Q(:,2).\hat{}2))\\
&Q(:,3) = Q(:,3)./\mathit{sqrt}(\mathit{sum}(Q(:,3).\hat{}2))\\
&\mathit{transQ} = Q.'\\
&\mathit{transT} = T.'\\
&\mathit{OUTPUT}\ : \mathit{SqrEps},\ \mathit{InvSrtEps},\ Q,\ \mathit{transQ} = Q.', T, \mathit{transT} = T.'
\end{aligned}
$$

Here $\mathit{Eps} = \mathcal{E}$, $\mathit{SqrEps} = \mathcal{E}^{\frac{1}{2}}$, $\mathit{InvSqrEps} = \mathcal{E}^{-\frac{1}{2}}$, $Q = \mathcal{Q}(\nu)$, $\mathit{transQ} = \mathcal{Q}^T(\nu)$, $T = \mathcal{T}(\nu)$, $\mathit{transT} = \mathcal{T}^T(\nu)$.

Let $\mathcal{E}$ be given, for example, as $\mathcal{E} = \mathrm{diag}(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33})$, where $\varepsilon_{11}, \varepsilon_{33}$ are symbols. Applying the standard commands we input matrices $\mathcal{E}, S(\nu)$ whose elements are symbols.

These commands are the following:

$$
\begin{aligned}
&eps11 = sym('eps11', 'real')\\
&eps33 = sym('eps33', 'real')\\
&\nu 1 = sym('\nu 1', 'real')\\
&\nu 2 = sym('\nu 2', 'real')\\
&\nu 3 = sym('\nu 3', 'real')\\
&Eps = [eps11, 0, 0; 0, eps11, 0; 0, 0, eps33]\\
&S = [\nu 2\hat{}2 + \nu 3\hat{}2, -\nu 1\hat{}2 * \nu 2\hat{}2, -\nu 1\hat{}2 * \nu 2\hat{}3;\\
&-\nu 1\hat{}2 * \nu 2\hat{}2, \nu 1\hat{}2 + \nu 3\hat{}2, -\nu 2\hat{}2 * \nu 3\hat{}2;\\
&-\nu 1\hat{}2 * \nu 3\hat{}2, -\nu 2\hat{}2 * \nu 3\hat{}2, \nu 1\hat{}2 + \nu 2\hat{}2].
\end{aligned}
$$

As a result of the above-described procedures we find

$$\mathcal{E}^{\frac{1}{2}} = \mathrm{diag}(\sqrt{\varepsilon_{11}}, \sqrt{\varepsilon_{11}}, \sqrt{\varepsilon_{33}}), \qquad \mathcal{E}^{-\frac{1}{2}} = \mathrm{diag}\left(\frac{1}{\sqrt{\varepsilon_{11}}}, \frac{1}{\sqrt{\varepsilon_{11}}}, \frac{1}{\sqrt{\varepsilon_{33}}}\right),$$

$$\mathcal{D}(\nu) = \mathrm{diag}\left(0, \frac{|\nu|^2}{\varepsilon_{11}}, \frac{\varepsilon_{11}\nu_1^2 + \varepsilon_{11}\nu_2^2 + \varepsilon_{33}\nu_3^2}{\varepsilon_{11}\varepsilon_{33}}\right),$$

$$\mathcal{Q}(\nu) = \begin{pmatrix} (\varepsilon_{11}/\varepsilon_{33})^{1/2}c(\nu_1/\nu_3) & -b(\nu_2/\nu_1) & -(\varepsilon_{33}/\varepsilon_{11})^{1/2}d(\nu_1\nu_3)\big/\left(\nu_1^2+\nu_2^2\right) \\ (\varepsilon_{11}/\varepsilon_{33})^{1/2}c(\nu_2/\nu_3) & b & -(\varepsilon_{33}/\varepsilon_{11})^{1/2}d(\nu_2\nu_3)\big/\left(\nu_1^2+\nu_2^2\right) \\ c & 0 & d \end{pmatrix},$$

$$\mathcal{T}(\nu) = \begin{pmatrix} (1/\varepsilon_{33})^{1/2}c(\nu_1/\nu_3) & -(1/\varepsilon_{11})^{1/2}b(\nu_2/\nu_1) & -\left(\varepsilon_{33}^{1/2}/\varepsilon_{11}\right)k(\nu_1\nu_3)/\left(\nu_1^2+\nu_2^2\right) \\ (1/\varepsilon_{33})^{1/2}c(\nu_2/\nu_3) & (1/\varepsilon_{11})^{1/2}b & -\left(\varepsilon_{33}^{1/2}/\varepsilon_{11}\right)k(\nu_2\nu_3)/\left(\nu_1^2+\nu_2^2\right) \\ (1/\varepsilon_{33})^{1/2}c & 0 & (1/\varepsilon_{33})^{1/2}k \end{pmatrix},$$

where

$$k = \left(\frac{\varepsilon_{11}\left(\nu_1^2+\nu_2^2\right) + \varepsilon_{33}\nu_3^2}{\varepsilon_{11}\left(\nu_1^2+\nu_2^2\right)}\right)^{-1/2}, \qquad c = \left(1 + \frac{\varepsilon_{11}}{\varepsilon_{33}}\frac{\nu_1^2}{\nu_3^2} + \frac{\varepsilon_{11}}{\varepsilon_{33}}\frac{\nu_2^2}{\nu_3^2}\right)^{-1/2}$$

$$d = \left(1 + \frac{\varepsilon_{33}}{\varepsilon_{11}}\frac{\nu_1^2\nu_3^2}{\left(\nu_1^2+\nu_2^2\right)^2} + \frac{\varepsilon_{33}}{\varepsilon_{11}}\frac{\nu_2^2\nu_3^2}{\left(\nu_1^2+\nu_2^2\right)^2}\right)^{-1/2},$$

$$b = \left(1 + \nu_2^2/\nu_1^2\right)^{-1/2}, \qquad |\nu|^2 = \nu_1^2 + \nu_2^2 + \nu_3^2.$$

The matrices $\mathcal{Q}^T(\nu)$, $\mathcal{T}^T(\nu)$ are defined as transpose to $\mathcal{Q}(\nu)$, $\mathcal{T}(\nu)$, respectively.

Remark 4. *If the matrix $\mathcal{E}$ is symmetric with more general structure then the matrices $\mathcal{E}^{\frac{1}{2}}$, $\mathcal{E}^{-\frac{1}{2}}$, $\mathcal{Q}(\nu)$, $\mathcal{Q}^T(\nu)$, $\mathcal{D}(\nu)$, $\mathcal{T}(\nu)$, $\mathcal{T}^T(\nu)$ can be found by described procedures explicitly but in the form of cumbersome symbolic expressions.*

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